

SOME RESULTS OF THE MARIÑO-VAFA FORMULA

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ABSTRACT. In this paper we derive some new Hodge integral identities by taking the limits of Mariño-Vafa formula. These identities include the formula of $\lambda_1 \lambda_g$ -integral on $\overline{\mathcal{M}}_{g,1}$, the vanishing result of $\lambda_g \text{ch}_{2l}(\mathbb{E})$ -integral on $\overline{\mathcal{M}}_{g,1}$ for $1 \leq l \leq g-3$. Using the differential equation of Hodge integrals, we give a recursion formula of λ_{g-1} -integrals. Finally, we give two simple proofs of λ_g conjecture and some examples of low genus integral.

1. INTRODUCTION

Based on string duality, Mariño and Vafa [10] conjectured a closed formula on certain Hodge integrals in terms of representations of symmetric groups. Recently, C.C. Liu, K. Liu and J. Zhou [6] proved this formula and derived some consequences from it [7]. In this paper we follow their method to derive some new Hodge integral identities. One of the main results of this paper is the following identity: if $1 \leq m \leq 2g-3$, then

$$\begin{aligned}
 (1.1) \quad & -(2g-2-m)!(-1)^{2g-3-m} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \text{ch}_{2g-2-m}(\mathbb{E}) \psi_1^m \\
 &= b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m}}{2g-1-k} B_{2g-1-m} \\
 &+ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, m-1)} \frac{(-1)^{2g_2-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2g-1-m}}{2g-1-k} B_{2g-1-m}.
 \end{aligned}$$

As a consequence, we find a new Hodge integral identity: if $g \geq 2$, then

$$(1.2) \quad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} [g(2g-3)b_g + b_1 b_{g-1}],$$

and also a vanishing result: if $g \geq 2$, then for any $1 \leq t \leq g-1$, we have

$$(1.3) \quad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \text{ch}_{2t}(\mathbb{E}) \psi_1^{2(g-1-t)} = 0.$$

Recently, Liu-Xu [8] derived a generalized formula for Hodge integrals of type (2) by using the λ_g conjecture.

The rest of this paper is organized as follows: In Section 2, we recall the Mariño-Vafa formula and the Mumford's relations. In Section 3, we prove our main theorem and derive a new Hodge integral identity. In Section 4, we give another simple proof of λ_g conjecture. In Section 5, we derive a recursion formula of λ_{g-1} -integrals. In the last section, we list some low genus examples.

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2. PRELIMINARIES

2.1. Partitions. A partition μ of a positive integer d is a sequence of integers $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{l(\mu)} > 0$ such that

$$\mu_1 + \dots + \mu_{l(\mu)} = d = |\mu|,$$

for each positive integer i , let

$$m_i(\mu) = |\{j | \mu_j = i, 1 \leq j \leq l(\mu)\}|.$$

The automorphism group $\text{Aut}(\mu)$ of μ consists of possible permutations among the μ_i 's, hence its order is given by

$$|\text{Aut}(\mu)| = \prod_i m_i(\mu)!,$$

define the numbers

$$\kappa_\mu = \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1), \quad z_\mu = \prod_j m_j(\mu)! j^{m_j(\mu)}.$$

The Young diagram of μ has $l(\mu)$ rows of adjacent squares: the i -th row has μ_i squares. The diagram of μ can be defines as the set of points $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $1 \leq j \leq \mu_i$, the conjugate of a partition μ is the partition μ' whose diagram is the transpose of the diagram μ . Finally, we introduce the hook length of μ at the squire $x \in (i, j)$:

$$h(x) = \mu_i + \mu'_j - i - j + 1.$$

Each partition μ of d corresponds to a conjugacy class $C(\mu)$ of the symmetric group S_d and each partition ν corresponds to an irreducible representation R_ν of S_d , let $\chi_\nu(C(\mu)) = \chi_{R_\nu}(C(\mu))$ be the value of the character χ_{R_ν} on the conjugacy class $C(\mu)$.

2.2. Mariño-Vafa formula. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \dots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \omega_C)$, the complex vector space of holomorphic one forms on C . Let $s_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$ denote the section of π which corresponds to the i -th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \dots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T_{x_i}^* C$ at the i -th marked point x_i . Consider the Hodge integral

$$(2.1) \quad \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \dots \psi_n^{j_n} \lambda_1^{k_1} \dots \lambda_g^{k_g}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of \mathbb{L}_i , and $\lambda_j = c_j(\mathbb{E})$ is the j -th Chern class of \mathbb{E} . The dimension of $\overline{\mathcal{M}}_{g,n}$ is $3g - 3 + n$, hence (4) is equal to zero unless $\sum_{i=1}^n j_i + \sum_{i=1}^g i k_i = 3g - 3 + n$. Let

$$(2.2) \quad \Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \dots + (-1)^g \lambda_g = \sum_{i=0}^g (-1)^i \lambda_i u^{g-i}$$

be the Chern polynomial of the dual bundle \mathbb{E}^\vee of \mathbb{E} . For any partition $\mu : \mu_1 \geq \mu_2 \geq \dots \geq \mu_{l(\mu)} > 0$, define

$$(2.3) \quad \mathcal{C}_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\text{Aut}(\mu)|} [\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \\ \cdot \int_{\mathcal{M}_{g,n}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau-1) \Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)},$$

$$(2.4) \quad \mathcal{C}_\mu(\lambda; \tau) = \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} \mathcal{C}_{g,\mu}(\tau),$$

here τ is a formal variable. Note that

$$\int_{\mathcal{M}_{g,n}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau-1) \Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \dots, p_n, \dots)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for a partition μ . Define generating functions

$$(2.5) \quad \mathcal{C}(\lambda; \tau, p) = \sum_{|\mu| \geq 1} \mathcal{C}_\mu(\lambda; \tau) p_\mu,$$

$$(2.6) \quad \mathcal{C}(\lambda; \tau, p)^\bullet = e^{\mathcal{C}(\lambda; \tau, p)}.$$

In [6], Chiu-chu Melissa Liu, Kefeng Liu and Jian Zhou have proved the following formula which was conjectured by Mariño and Vafa in [10].

Theorem 2.1. (Mariño-Vafa Formula) *For every partition μ , we have*

$$\mathcal{C}(\lambda; \tau, p) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\mu} \left(\sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n \sum_{|\nu^i| = |\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau + \frac{1}{2}) \kappa_{\nu^i} \lambda / 2} V_{\nu^i}(\lambda) \right) p_\mu, \\ \mathcal{C}(\lambda; \tau, p)^\bullet = \sum_{|\mu| \geq 0} \left(\sum_{|\nu| = |\mu|} \frac{\chi_\nu(C(\mu))}{z_\mu} e^{\sqrt{-1}(\tau + \frac{1}{2}) \kappa_\nu \lambda / 2} V_\nu(\lambda) \right) p_\mu,$$

where

$$V_\nu(\lambda) = \prod_{1 \leq a < b \leq l(\nu)} \frac{\sin[(\nu_a - \nu_b + b - a)\lambda/2]}{\sin[(b - a)\lambda/2]} \frac{1}{\prod_{i=1}^{l(\nu)} \prod_{v=1}^{\nu_i} 2 \sin[(v - i + l(\nu))\lambda/2]}.$$

It is known that

$$V_\nu(\lambda) = \frac{1}{2^{l(\nu)} \prod_{x \in \nu} \sin[h(x)\lambda/2]}.$$

2.3. Mumford's relations. Let $c_t(\mathbb{E}) = \sum_{i=0}^g t^i \lambda_i$, then we have

$$c_{-t}(\mathbb{E}) = t^g \Lambda_g^\vee \left(\frac{1}{t} \right).$$

Mumford's relations [11] are given by

$$(2.7) \quad c_t(\mathbb{E}) c_{-t}(\mathbb{E}) = 1,$$

equivalently

$$(2.8) \quad \Lambda_g^\vee(t) \Lambda_g^\vee(-t) = (-1)^g t^{2g},$$

then

$$(2.9) \quad \lambda_k^2 = \sum_{i=1}^k (-1)^{i+1} 2\lambda_{k-i} \lambda_{k+i},$$

where $\lambda_0 = 1$ and $\lambda_k = 0$ for $k > g$. Let x_1, \dots, x_g be the formal Chern roots of \mathbb{E} , the Chern character is defined by

$$\text{ch}(\mathbb{E}) = \sum_{i=1}^g e^{x_i} = g + \sum_{n=1}^{+\infty} \sum_{i=1}^g \frac{x_i^n}{n!},$$

we write

$$(2.10) \quad \text{ch}_0(\mathbb{E}) = g,$$

$$(2.11) \quad \text{ch}_n(\mathbb{E}) = \frac{1}{n!} \sum_{i=1}^g x_i^n, \quad n = 1, 2, \dots.$$

From the above identities we have the relation between $\text{ch}_n(\mathbb{E})$ and λ_n :

$$(2.12) \quad n! \text{ch}_n(\mathbb{E}) = \sum_{i+j=n} (-1)^{i-1} i \lambda_i \lambda_j, \quad n < 2g,$$

$$(2.13) \quad \text{ch}_n(\mathbb{E}) = 0, \quad n \geq 2g.$$

It is easy to see that

$$\begin{aligned} (2g-1)! \text{ch}_{2g-1}(\mathbb{E}) &= (-1)^{g-1} \lambda_{g-1} \lambda_g, \\ (2g-2)! \text{ch}_{2g-2}(\mathbb{E}) &= (-1)^{g-1} ((2g-2) \lambda_{g-2} \lambda_g - (g-1) \lambda_{g-1}^2), \\ (2g-3)! \text{ch}_{2g-3}(\mathbb{E}) &= (-1)^{g-1} (3 \lambda_{g-3} \lambda_g - \lambda_{g-1} \lambda_{g-2}). \end{aligned}$$

2.4. Bernoulli numbers. The Bernoulli numbers B_m are defined by the following series expansion:

$$(2.14) \quad \frac{t}{e^t - 1} = \sum_{m=0}^{+\infty} B_m \frac{t^m}{m!},$$

the first few terms are given by

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}.$$

Finally we recall two formulas which will be used later:

$$(2.15) \quad \frac{t/2}{\sin(t/2)} = 1 + \sum_{g=1}^{+\infty} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} t^{2g},$$

$$(2.16) \quad \sum_{i=1}^{d-1} i^m = \sum_{k=0}^m \frac{\binom{m+1}{k}}{m+1} B_k d^{m+1-k},$$

where m is a positive integer.

3. SOME NEW RESULTS FROM MARIÑO-VAFA FORMULA

In this section we derive some new results from the Mariño-Vafa formula, we will need two formulas in [7, 2.1 and 5.1].

Theorem 3.1. *We have the following results:*

$$(3.1) \quad \sum_{g \geq 0} \lambda^{2g} \int_{\mathcal{M}_{g,1}} \frac{\frac{d}{d\tau}|_{\tau=0} [\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau-1)]}{1 - d\psi_1} \\ = - \sum_{a=1}^{d-1} \frac{1}{a} \frac{d\lambda/2}{d\sin(d\lambda/2)} + \sum_{i+j=d, i,j \neq 0} \frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)},$$

$$(3.2) \quad \frac{d}{d\tau} \Big|_{\tau=0} [\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau-1)] = -\lambda_{g-1} - \lambda_g \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}).$$

3.1. The coefficient of λ^{2g} . Introduce the notation

$$b_g = \begin{cases} 1, & g = 0, \\ \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0, \end{cases}$$

then the coefficient of λ^{2g} in $-\sum_{a=1}^{d-1} \frac{1}{a} \frac{d\lambda/2}{d\sin(d\lambda/2)}$ is

$$(3.3) \quad \left(- \sum_{a=1}^{d-1} \frac{1}{a} \right) \cdot b_g d^{2g-1}.$$

If $g_1, g_2 \geq 0$ and $g_1 + g_2 = g$, define

$$(3.4) \quad F_{g_1, g_2}(d) = \sum_{i+j=d, i,j \neq 0} i^{2g_1-1} j^{2g_2-1}.$$

In [6] it is showed that if $g_1, g_2 \geq 1$, then

$$(3.5) \quad F_{g_1, g_2}(d) = \sum_{k=0}^{2g_2-1} \sum_{l=0}^{2g_2-2-k} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{l} B_l d^{2g-1-l},$$

for the rest case we have

$$\begin{aligned}
(3.6) \quad F_{0,g}(d) &= \sum_{i+j=d, i,j \neq 0} i^{-1} j^{2g-1} \\
&= \sum_{i=1}^{d-1} i^{-1} (d-i)^{2g-1} \\
&= \sum_{k=0}^{2g-1} (-1)^{2g-1-k} \binom{2g-1}{k} d^k \sum_{i=1}^{d-1} i^{2g-2-k} \\
&= \sum_{k=0}^{2g-3} (-1)^{2g-1-k} \binom{2g-1}{k} d^k \sum_{l=0}^{2g-k-2} \frac{\binom{2g-1-k}{l}}{2g-1-k} B_l d^{2g-1-k-l} \\
&\quad - (2g-1) d^{2g-2} (d-1) + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i} \\
&= \sum_{k=0}^{2g-2} \sum_{l=0}^{2g-k-2} \binom{2g-1}{k} \binom{2g-1-k}{l} \frac{(-1)^{2g-1-k}}{2g-1-k} B_l d^{2g-1-l} \\
&\quad + (2g-1) d^{2g-2} + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i}.
\end{aligned}$$

Note that $F_{0,g}(d) = F_{g,0}(d)$ and

$$(3.7) \quad \sum_{i+j=d, i,j \neq 0} \frac{\lambda^2}{8 \sin(i\lambda/2) \sin(j\lambda/2)} = \frac{1}{2} \sum_{g \geq 0} \lambda^{2g} \left(\sum_{g_1+g_2=g} b_{g_1} b_{g_2} F_{g_1,g_2}(d) \right).$$

3.2. The Main Theorem. Let

$$\sum_{g \geq 0} \lambda^{2g} LHS = \sum_{g \geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\frac{d}{d\tau} \big|_{\tau=0} [\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau-1)]}{1 - d\psi_1},$$

and

$$\sum_{g \geq 0} \lambda^{2g} RHS = - \sum_{a=1}^{d-1} \frac{1}{a} \frac{d\lambda/2}{d \sin(d\lambda/2)} + \sum_{i+j=d, i,j \neq 0} \frac{\lambda^2}{8 \sin(i\lambda/2) \sin(j\lambda/2)},$$

then we have

$$\begin{aligned}
(3.8) \quad LHS &= - \int_{\overline{\mathcal{M}}_{g,1}} (\lambda_{g-1} \psi_1^{2g-1}) d^{2g-1} \\
&\quad - \sum_{k=0}^{2g-2} \left[(2g-2-k)! (-1)^{2g-3-k} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \text{ch}_{2g-2-k}(\mathbb{E}) \psi_1^k \right] d^k,
\end{aligned}$$

$$(3.9) \quad RHS = - \sum_{a=1}^{d-1} \frac{b_g}{a} d^{2g-1} + b_g F_{0,g}(d) + \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} F_{g_1,g_2}(d).$$

Hence we can derive our main theorem:

Theorem 3.2. *If $1 \leq m \leq 2g - 3$ and $g \geq 2$, then*

$$\begin{aligned} & -(2g - 2 - m)!(-1)^{2g-3-m} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \text{ch}_{2g-2-m}(\mathbb{E}) \psi_1^m \\ &= b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m} \\ &+ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, m-1)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m}. \end{aligned}$$

Remark 3.3. Liu-Liu-Zhou [7] have only considered the cases $m = 2g - 1$ and $m = 1$.

3.3. The case of $m = 2g - 3$. If $m = 2g - 3$, we find that $1! \text{ch}_1(\mathbb{E}) = \lambda_1$, then

$$\begin{aligned} LHS &= - \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \lambda_1 \psi_1^{2g-3}, \\ RHS &= b_g \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2} B_2 \\ &+ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2} B_2. \end{aligned}$$

From the above formula we obtain a new result of the Hodge integral.

Theorem 3.4. *If $g \geq 2$, then*

$$(3.10) \quad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} [g(2g-3)b_g + b_1 b_{g-1}].$$

Proof. Note that

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = -b_g B_2 \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2} \\ & - \frac{B_2}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2}, \end{aligned}$$

let us write

$$\begin{aligned} A_1 &= \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2}, \\ f_1(x) &= \sum_{k=0}^{2g-3} (-1)^{2g-1-k} \binom{2g-1}{k} (2g-2-k) x^{2g-3-k}, \\ g_1(x) &= \sum_{k=0}^{2g-2} (-1)^{2g-1-k} \binom{2g-1}{k} x^{2g-2-k}, \end{aligned}$$

then

$$A_1 = \frac{1}{2} \sum_{k=0}^{2g-4} (-1)^{2g-1-k} \binom{2g-1}{k} (2g-2-k), \quad xg_1(x) = (1-x)^{2g-1} - 1, \quad f_1(x) = g'_1(x).$$

Hence

$$f_1(x) = \frac{(2g-1)x(1-x)^{2g-2} - (1-x)^{2g-1} + 1}{x^2}, \quad f_1(1) = 1,$$

and we obtain

$$A_1 = \frac{1}{2} \left[f_1(1) - \binom{2g-1}{2g-3} \right] = -\frac{1}{2} \left[\binom{2g-1}{2g-3} - 1 \right].$$

Similarly, we write

$$A_2 = \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2},$$

then

$$A_2 = \begin{cases} \frac{1}{2} \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} (2g-2-k) \binom{2g_2-1}{k}, & g_2 \leq g-2, \\ \frac{1}{2} \sum_{k=0}^{2g-4} (-1)^{2g-1-k} (2g-2-k) \binom{2g-3}{k}, & g_2 = g-1. \end{cases}$$

Case 1: $g_2 \geq g-2$. Let

$$\begin{aligned} f_2(x) &= \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} (2g-2-k) \binom{2g_2-1}{k} x^{2g-3-k}, \\ g_2(x) &= \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} \binom{2g_2-1}{k} x^{2g-2-k}. \end{aligned}$$

Since $g \geq g_2 + 2$, then $2g-3-(2g_2-1) \geq 2 > 0$ and $g'_2(x) = f_2(x)$. On the other hand

$$g_2(x) = (1-x)^{2g_2-1} x^{2g_1-1},$$

hence

$$g'_2(x) = -(2g_2-1)(1-x)^{2g_2-2} x^{2g_1-1} + (2g_1-1)(1-x)^{2g_2-1} x^{2g_1-2}$$

and

$$f_2(1) = \begin{cases} -1, & g_2 = 1, \\ 0, & 1 < g_2 \leq g-2. \end{cases}$$

Case 2: $g_2 = g-1$. let

$$\begin{aligned} f_3(x) &= \sum_{k=0}^{2g-4} (-1)^{2g-1-k} (2g-2-k) \binom{2g-3}{k} x^{2g-3-k}, \\ g_3(x) &= \sum_{k=0}^{2g-4} (-1)^{2g-1-k} \binom{2g-3}{k} x^{2g-2-k}. \end{aligned}$$

Since $2g-3-(2g-4) = 1 > 0$,

$$\frac{g_3(x)}{x} = \sum_{k=0}^{2g-4} (-1)^{2g-3-k} \binom{2g-3}{k} x^{2g-3-k} = (1-x)^{2g-3} - 1$$

and

$$g'_3(x) = (1-x)^{2g-3} - 1 - (2g-3)(1-x)^{2g-4}x,$$

therefore we have

$$f_3(1) = \begin{cases} -2, & g = 2, \\ -1, & g > 2. \end{cases}$$

From the values of $f_1(1), f_2(1), f_3(1)$, we obtain

$$\begin{aligned} & \int_{\mathcal{M}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} \\ &= -b_g B_2 A_1 - \frac{B_2}{2} \left[\frac{1}{2} \sum_{g_1+g_2=g, 1 \leq g_2 \leq g-2} b_{g_1} b_{g_2} f_2(1) + \frac{1}{2} b_1 b_{g-1} f_3(1) \right] \\ &= \frac{B_2}{2} [-b_g A_1 + b_1 b_{g-1}] \\ &= \frac{1}{12} [g(2g-3)b_g + b_1 b_{g-1}] \end{aligned}$$

□

Since $B_n = 0$ for n odd and $n > 1$, we also have the following vanishing result.

Theorem 3.5. *If $g \geq 2$, then for any $1 \leq t \leq g-1$, we have*

$$(3.11) \quad \int_{\mathcal{M}_{g,1}} \lambda_g \text{ch}_{2t}(\mathbb{E}) \psi_1^{2(g-1-t)} = 0.$$

4. ANOTHER SIMPLE PROOF OF λ_g CONJECTURE

Let $|\mu| = d, l(\mu) = n$, denote by $[\mathcal{C}_{g,\mu}(\tau)]_k$ the coefficient of τ^k in the polynomial $\mathcal{C}_{g,\mu}(\tau)$, and let

$$\begin{aligned} J_{g,\mu}^0(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)} \mathcal{C}_{g,\mu}(\tau), \\ J_{g,\mu}^1(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)-1} \left(\sum_{\nu \in J(\mu)} I_1(\nu) \mathcal{C}_{g,\nu}(\tau) + \sum_{\nu \in C(\mu)} I_2(\nu) \mathcal{C}_{g-1,\nu}(\tau) \right. \\ &\quad \left. + \sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) \mathcal{C}_{g_1,\nu^1}(\tau) \mathcal{C}_{g_2,\nu^2}(\tau) \right). \end{aligned}$$

The set $J(\mu)$ consists of partitions of d of the form

$$\nu = (\mu_1, \dots, \widehat{\mu}_i, \dots, \mu_{l(\mu)}, \mu_i + \mu_j)$$

and the set $C(\mu)$ consists of partitions of d of the form

$$\nu = (\mu_1, \dots, \widehat{\mu}_i, \dots, \mu_{l(\mu)}, j, k)$$

where $j + k = \mu_i$. The definitions of I_1, I_2 and I_3 can be found in [5]. Liu-Liu-Zhou [6] have proved the following differential equation:

$$(4.1) \quad \frac{d}{d\tau} J_{g,\mu}^0(\tau) = -J_{g,\mu}^1(\tau).$$

It is straightforward to check that

$$\begin{aligned} [\mathcal{C}_{g,\mu}(\tau)]_{n-1} &= -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)}, \\ \left[\sum_{\nu \in J(\mu)} I_1(\nu) \mathcal{C}_{g,\nu}(\tau) \right]_{n-2} &= -\frac{\sqrt{-1}^{d+n-1}}{|\text{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1} (1 - \mu_i \psi_i)}, \end{aligned}$$

$$\begin{aligned} \left[\sum_{\nu \in C(\mu)} I_2(\nu) \mathcal{C}_{g-1,\nu}(\tau) \right]_{n-2} &= 0, \\ \left[\sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) \mathcal{C}_{g_1,\nu^1}(\tau) \mathcal{C}_{g_2,\nu^2}(\tau) \right]_{n-2} &= 0, \end{aligned}$$

hence, from (31) we have the identity

$$(4.2) \quad \frac{n-1}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} = \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)}.$$

Theorem 4.1. *For any partition $\mu : \mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0$ of d and $g > 0$, then*

$$(4.3) \quad \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} = d^{2g+n-3} b_g.$$

Proof. Recall the definition of $I_1(\nu)$, where $\nu = (\mu_1, \dots, \widehat{\mu}_i, \dots, \widehat{\mu}_j, \dots, \mu_n, \mu_i + \mu_j)$:

$$I_1(\nu) = \frac{\mu_i + \mu_j}{1 + \delta_{\mu_j}^{\mu_i}} m_{\mu_i + \mu_j}(\nu),$$

and it is easy to see that

$$\frac{m_{\mu_i + \mu_j}(\nu)}{|\text{Aut}(\nu)|} = \frac{m_{\mu_i}(\mu)(m_{\mu_j}(\mu) - \delta_{\mu_j}^{\mu_i})}{|\text{Aut}(\mu)|}.$$

Let

$$\mu : \underbrace{\mu_{k_1} = \dots = \mu_{k_1}}_{t_1} > \underbrace{\mu_{k_2} = \dots = \mu_{k_2}}_{t_2} > \dots > \underbrace{\mu_{k_s} = \dots = \mu_{k_s}}_{t_s} > 0,$$

where

$$\sum_{i=1}^s t_i = n, \quad \sum_{i=1}^s t_i \mu_{k_i} = d,$$

then

$$\begin{aligned}
& \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \\
&= \frac{1}{|\text{Aut}(\mu)|} \sum_{\nu \in J(\mu)} \frac{\mu_i + \mu_j}{1 + \delta_{\mu_j}^{\mu_i}} [m_{\mu_i}(\mu)(m_{\mu_j}(\mu) - \delta_{\mu_j}^{\mu_i})] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[\frac{1}{2} \sum_{i=1}^s \sum_{j \neq i} (\mu_{k_i} + \mu_{k_j}) m_{\mu_{k_i}}(\mu) m_{\mu_{k_j}}(\mu) + \sum_{i=1}^s \mu_{k_i} m_{\mu_{k_i}}(\mu) (m_{\mu_{k_i}}(\mu) - 1) \right] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[\frac{1}{2} \sum_{i=1}^s \sum_{j \neq i} (\mu_{k_i} + \mu_{k_j}) t_i t_j + \sum_{i=1}^s \mu_{k_i} t_i (t_i - 1) \right] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[\sum_{i=1}^s \sum_{j \neq i} \mu_{k_j} t_j t_i + \sum_{i=1}^s \mu_{k_i} t_i^2 - d \right] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[\sum_{i=1}^s t_i (d - \mu_{k_i} t_i) + \sum_{i=1}^s \mu_{k_i} t_i^2 - d \right] \\
&= \frac{(n-1)d}{|\text{Aut}(\mu)|}.
\end{aligned}$$

By the induction of n and the initial value of the Mariño-Vafa formula

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_g}{1 - \mu_1 \psi_1} = d^{2g-2} b_g,$$

we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} = d \cdot d^{2g+n-1-3} b_g = d^{2g+n-3} b_g.$$

□

Corollary 4.2. *The following λ_g conjecture is true:*

$$(4.4) \quad \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} = \binom{2g+n-3}{k_1, \dots, k_n} b_g,$$

where $g > 0$.

5. A RECURSION FORMULA OF THE λ_{g-1} INTEGRAL

E.Getzler, A.Okounkov and R.Pandharipande have derived explicit formula for the multipoint series of \mathbb{CP}^1 in degree 0 from the Toda hierarchy [2], then they obtained certain formulas for the Hodge integrals $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{g-1} \psi_1^{k_1} \cdots \psi_n^{k_n}$. In this section we give an effective recursion formula of the λ_{g-1} integrals using Mariño-Vafa formula. It is straightforward to check the following lemma.

Lemma 5.1. *We have the following identities*

$$\begin{aligned}
\left[\prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \right]_0 &= 1, \\
\left[\prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \right]_1 &= \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a}, \\
[\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau - 1) \Lambda_g^\vee(\tau)]_0 &= \lambda_g, \\
[\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau - 1) \Lambda_g^\vee(\tau)]_0 &= -\lambda_{g-1} - \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g,
\end{aligned}$$

and

$$\begin{aligned}
[\mathcal{C}_{g,\mu}(\tau)]_{n-1} &= -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)}, \\
[\mathcal{C}_{g,\mu}(\tau)]_n &= -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \left[n - 1 + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right] \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\
&\quad + \frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)}.
\end{aligned}$$

Now, we can state our main theorem in this section using equation (31).

Theorem 5.2. *For any partition μ with $l(\mu) = n$, we have the following recursion formula*

$$\begin{aligned}
&\frac{n}{|\text{Aut}(\mu)|} \left[n - 1 + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right] \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\
&- \frac{n}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\
&= \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \left[n - 2 + \sum_{i=1}^{n-1} \sum_{a=1}^{\nu_i-1} \frac{\nu_i}{a} \right] \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)} \\
&- \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)} \\
&+ \sum_{g_1+g_2=g, g_1, g_2 \geq 0} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} \frac{I_3(\nu^1, \nu^2)}{|\text{Aut}(\nu^1)| |\text{Aut}(\nu^2)|} \\
&\cdot \int_{\overline{\mathcal{M}}_{g_1, n_1}} \frac{\lambda_{g_1}}{\prod_{i=1}^{n_1} (1 - \nu_i^1 \psi_i)} \int_{\overline{\mathcal{M}}_{g_2, n_2}} \frac{\lambda_{g_2}}{\prod_{i=1}^{n_2} (1 - \nu_i^2 \psi_i)}.
\end{aligned}$$

5.1. The λ_g -Integral. In this subsection, we re-derive the λ_g -integral from theorem 5.2.. Let $\mu_i = N x_i$ for some $N \in \mathbb{N}$ and $x_i \in \mathbb{R}$, from Kim-Liu[4]'s method and consider the

coefficients of $\ln N N^{2g+n-2}$ in theorem 5.2., then

$$\begin{aligned}
& n(x_1 + \cdots + x_n) \prod_{l=1}^n x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (x_i + x_j)^{k_i+k_j} (x_1 + \cdots + x_n) \prod_{l \neq i,j} x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n-1}} \lambda_g \psi^{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l} \\
&+ (x_1 + \cdots + x_n) \prod_{l=1}^n x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l},
\end{aligned}$$

i.e.

$$\begin{aligned}
(5.1) \quad & (n-1) \prod_{l=1}^n x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (x_i + x_j)^{k_i+k_j} \prod_{l \neq i,j} x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n-1}} \lambda_g \psi^{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l}.
\end{aligned}$$

After introducing the formal variables $s_l \in \mathbb{R}^+$ and applying the Laplace transformation

$$\int_0^{+\infty} x^k e^{-x/2s} dx = k!(2s)^{k+1}, \quad s > 0,$$

we select the coefficient of $\prod_{l=1}^n (2s_l)^{k_l+1}$ from the transformation of (35), then we derive

$$(5.2) \quad (n-1) \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} = \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \frac{(k_i + k_j)!}{k_i! k_j!} \int_{\overline{\mathcal{M}}_{g,n-1}} \lambda_g \psi^{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l}.$$

By the induction of n , we obtain the λ_g conjecture

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} = \binom{2g+n-3}{k_1, \dots, k_n} b_g,$$

in fact, in (36) we have

$$\begin{aligned}
RHS &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \frac{(k_i + k_j)!}{k_i! k_j!} \frac{(2g+n-4)!}{\prod_{l \neq i,j} k_l! (k_i + k_j - 1)!} b_g \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \frac{k_i + k_j}{2g+n-3} \binom{2g+n-3}{k_1, \dots, k_n} b_g,
\end{aligned}$$

note that $k_1 + \cdots + k_n = 2g + n - 3$, therefore

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (k_i + k_j) &= \frac{1}{2} \sum_{i=1}^n [(n-1)k_i + (2g+n-3-k_i)] \\
&= \frac{1}{2} [(n-2)(2g+n-3) + (2g+n-3)n] \\
&= \frac{1}{2} [(2n-2)(2g+n-3)] \\
&= (n-1)(2g+n-3).
\end{aligned}$$

5.2. The Recursion Formula of λ_{g-1} -integral. We have found the *singular part* $\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a}$ in theorem 5.2., using the following theorem, we can eliminate this part and derive the recursion formula of λ_{g-1} -integral. The notation $[F]_{sing}$ means the singular part of F . First, in theorem 5.2., we have

$$\begin{aligned} \left[\frac{LHS}{d^{2g+n-4}b_g} \right]_{sing} &= n \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d, \\ \left[\frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \left[\sum_{l \neq i, j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} + (\mu_i + \mu_j) \sum_{a=1}^{\mu_i+\mu_j-1} \frac{1}{a} \right] \\ &\quad + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a}. \end{aligned}$$

Theorem 5.3. *Under the above notation, we have*

$$\left[\frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} = \left[\frac{LHS}{d^{2g+n-4}b_g} \right]_{sing} + 2(n-1)d.$$

Proof. Since

$$\begin{aligned} &\sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a} \\ &= \sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_i(\mu_i + \mu_j)}{a} \\ &= \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j} \frac{(\mu_i + \mu_j)^2}{a} \\ &\quad - \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{\mu_i + \mu_j}{a} + \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j} \frac{(\mu_i + \mu_j)}{a} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} \\ &\quad + \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)}{a} - \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j), \end{aligned}$$

where we use the identity

$$\begin{aligned} \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)\mu_i}{a} &= \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)\mu_j}{a} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a}. \end{aligned}$$

Note that $\sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) = 2(n-1)d$, hence

$$\begin{aligned}
& \left[\frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d \\
&+ \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \mu_i \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)}{a} + \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \\
&= \left(\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} \\
&+ \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a} + 2(n-1)d,
\end{aligned}$$

it is straightforward to check that

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} &= (n-2) \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a}, \\
\sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a} &= \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} + \sum_{i=1}^n \sum_{j \neq i} \mu_j \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a}, \\
\sum_{i=1}^n \sum_{j \neq i} \mu_j \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} &= (n-1) \sum_{i=1}^n \mu_i \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
& \left[\frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} \\
&= \left(\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + (n-2) \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} \\
&+ \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} + (n-1) \sum_{i=1}^n \mu_i \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} + 2(n-1)d \\
&= \left(\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + (n-1) \sum_{i=1}^n \mu_i \left(\sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} + \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) + 2(n-1)d \\
&= \left(\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + (n-1) \left(\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + 2(n-1)d \\
&= \left[\frac{LHS}{d^{2g+n-4}b_g} \right]_{sing} + 2(n-1)d.
\end{aligned}$$

□

Let $\mathbb{R}^k[\mu_1, \dots, \mu_n]$ be the space of all homogeneous polynomials with real coefficients in μ_1, \dots, μ_n of degree k , then it is the subring of $\mathbb{R}[\mu_1, \dots, \mu_n]$. From the Theorem 5.3, we obtain the recursion formula of λ_{g-1} Hodge integral.

Theorem 5.4. *For any partition μ with $l(\mu) = n$ and $|\mu| = d$, we have the recursion formula*

$$\begin{aligned} & \frac{n}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_{g-1}}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\ = & \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_{g-1}}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)} \\ - & \sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} \frac{I_3(\nu^1, \nu^2)}{|\text{Aut}(\nu^1)| |\text{Aut}(\nu^2)|} d_1^{2g_1+n_1-3} d_2^{2g_2+n_2-3} b_{g_1} b_{g_2}. \end{aligned}$$

under the ring $\mathbb{R}^{2g-2+n}[\mu_1, \dots, \mu_n]$, where $l(\nu^i) = n_i$ and $|\nu^i| = d_i$ for $i = 1, 2$.

Remark 5.5. When we consider the simplest case $n = 1$, the above identity become the formula used in [6].

6. SOME EXAMPLES OF THE MAIN THEOREM

In this section we give some examples of theorem 3.2.

6.1. The case of $g = 3$. If $g = 3$, then $1 \leq m \leq 3$. We consider three cases.

6.1.1. $m=1$. $LHS = -3 \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \text{ch}_3(\mathbb{E}) \psi_1$, and $3 \text{ch}_3(\mathbb{E}) = \sum_{i+j=3} (-1)^{i-1} i \lambda_i \lambda_j = 3\lambda_3 - \lambda_1 \lambda_2$, then we get

$$\begin{aligned} LHS &= \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 (\lambda_1 \lambda_2 - 3\lambda_3) \psi_1 = \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_2 \lambda_3 \psi_1, \\ RHS &= b_3 \frac{(-1)^5}{5} \binom{5}{0} \binom{5}{4} B_4 + \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \frac{-1}{5} \binom{2g_2-1}{0} \binom{5}{4} B_4 \\ &= -B_4 (b_3 + b_1 b_2). \end{aligned}$$

Since $b_1 = \frac{1}{24}$, $b_2 = \frac{7}{5760}$, $b_3 = \frac{31}{967680}$, we have

$$(6.1) \quad \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_2 \lambda_3 \psi_1 = \frac{1}{362880}.$$

6.1.2. $m=2$. In this case we have $LHS = 2 \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \text{ch}_2(\mathbb{E}) \psi_1^2$, and $2! \text{ch}_2(\mathbb{E}) = 2\lambda_2 - \lambda_1^2$, $B_3 = 0$. Then we have

$$\begin{aligned} LHS &= \int_{\overline{\mathcal{M}}_{3,1}} (2\lambda_2\lambda_3 - \lambda_3\lambda_1^2) \psi_1^2, \\ RHS &= b_3 \sum_{k=0}^1 \frac{(-1)^{4-k}}{5-k} \binom{5}{k} \binom{5-k}{3} B_3 \\ &\quad + \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^1 \frac{(-1)^{5-k}}{5-k} \binom{2g_2-1}{k} \binom{5-k}{3} B_3 \\ &= 0, \end{aligned}$$

hence

$$\int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \lambda_1^2 \psi_1^2 = 2 \int_{\overline{\mathcal{M}}_{3,1}} \lambda_2 \lambda_3 \psi_1^2.$$

Using the formula $\int_{\overline{\mathcal{M}}_{3,1}} \lambda_2 \lambda_3 \psi_1^2 = \frac{1}{120960}$, we get

$$(6.2) \quad \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \lambda_1^2 \psi_1^2 = \frac{1}{60480}.$$

6.1.3. $m=3$. In this case $LHS = - \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \text{ch}_1(\mathbb{E}) \psi_1^3$ and $\text{ch}_1(\mathbb{E}) = \lambda_1$, hence

$$\begin{aligned} LHS &= - \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_3 \psi_1^3, \\ RHS &= b_3 \sum_{k=0}^2 \frac{(-1)^{5-k}}{5-k} \binom{5}{k} \binom{5-k}{2} B_2 \\ &\quad + \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2)} \frac{(-1)^{2g_2-1-k}}{5-k} \binom{2g_2-1}{k} \binom{5-k}{2} B_2 \\ &= -\frac{9}{2} b_3 B_2 - \frac{1}{2} b_1 b_2 B_2 \\ &= -\frac{41}{1451520}, \end{aligned}$$

so

$$(6.3) \quad \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_3 \psi_1^3 = \frac{41}{145120}.$$

Remark 6.1. The values of (37) and (39) match with the results in [9], the identity (38) is a new result.

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